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地方外部性模式之分析

陳心蘋[1]

摘要

Eeckhout (2004)的一般均衡理論可解釋吉伯特定理中的都市成長過程與分佈。該文對於模型中的主要驅動力之一的地方外部性，並無進一步探討。本研究的目的是分析地方外部性形式特質與都市成長過程及人口分佈的關係。本研究對模型中之地方外部性進行延伸分析，發現當地方外部性中的產出都市人口彈性固定時，一般均衡理論中的隨機生產過程可推導出比例成長的都市人口；同時，都市人口的上尾端分布會趨近於普瑞夫定理・此結果在Eeckhout (2004)中未提出。此外，地方外部性中的擁塞成本越大，估計的吉尼係數越小，都市間人口差異越小。該理論隱含當產出都市人口彈性為負時，技術衝擊越大都市規模越大；當擁擠成本主導淨地方外部性時，技術衝擊越大都市規模越大。

關鍵字：普瑞夫定理、產出的都市人口彈性、*JEL*R12、R23

1 國立政治大學經濟學系教授
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An Analysis of the Form of Local Externality

Hsin-Ping Chen

Department of Economics National Chengchi University Taipei, Taiwan.

Abstract

Ageneral equilibrium model proposed in Eeckhout (2004) explains Gibrat's law in growth process and size distribution of cities. One of the driving forces in the modelis local externality; however there is lack of further exploration of this key driving force. The purpose of this paper is to examine the feature and relation between local externality and the resulted growth process as well as size distribution; this is not discussed in Eeckhout (2004). This paper provides an extension of local externality and finds that the theory could generate proportionate growth of cities and Zipf's law in its tail only if the size elasticity of production in local externality is a constant. This result shows the condition of the theory in explaining the empirical size distribution of cities which is not examined in Eeckhout (2004). We also finds that an increase of congestion cost will lead to more evenly distributed cities.Moreover, the theory implies that larger technologyshocks lead to larger cities when the size elasticity of production is negative; larger technologyshocks lead to bigger cities if the congesting cost dominates the net local externalities.

Keywords: Zipf's law, Size elasticity of production, *JEL*R12, R23

1. Introduction

Zipf's law in the upper tail and proportionate growth of cities are two empirical regularities concerning the size distribution and growth process of cities.Further, itis verified empirically that size distribution of cities fit the lognormal distribution combined with Zipf's law in the upper tail.[2]The size distribution of cities is resulted from the evolution of cites,and the growth of cities is determined by the location decision of decision makers across cities.[3] The mobility of workers and firms is closely related to congestion cost.To understand the underlying mechanism of the formation of growth and size distribution of cities provide information for regional and transport policy planning.

Zipf's law provides a reasonable approximation for the upper tail of the distribution of size of cities which refers to population make it the regularity of city distribution. For any variable X, $P(X > x) = cx^{-\zeta}$, where ζ denotes the exponent of the Pareto distribution. Zipf's law corresponds to the statement that $\zeta = 1$. Gabaix (1999*a*) and Gabaix (1999*b*) explore a statistical process to explain Zipf's law. In the model, total population and the number of cities are both fixed. The population of city grows multiplicatively as proportionate growth process. The proportionate growth of city sizes which refers to city population lead to asymptotically lognormal distribution which is empirically shown to be reasonable approximation for the body of city size distribution.[4] Gabaix (1999*a*) derives a steady state tail distribution as Zipf's law given the proportionate growth of population. This work gives Zipf's law a statistical explanation.

Rossi-Hansberg and Wright (2007) apply multiplicative productivity shocks for an entire industry. There are no idiosyncratic productivity differences among cities as in Eeckhout (2004). Permanent industry shocks and temporary shocks affect factor accumulation. An alternative statistics model with additive process proposed by Simon (1955) assumes the aggregate population growth is discrete increments. The probability that a city grows is proportional to its population. This model lead to a Pareto distribution of city sizes. The exponent of Pareto distribution becomes one as the probability of new born cities equals zero.[5]Duranton and Puga (2014) show that adding multiplicative and cumulative random shocks to the productivity shifter of cities inRossi-Hansberg and Wright (2007) could generate lognormal size distribution of cities. Behrens, Duranton, and Robert-Nicoud (2014) propose a model of sorting across cities to generate Zipf's law. Hsu (2012) also applies the central place theory to generate Zip's law.

There are two types of urban growth models proposed in literatures to explain the growth

[2] Ioannides and Skouras (2013), González-Val et al. (2013), Giesen et al. (2012) ,Giesen et al. (2010) and Eeckhout (2004).
[3] George K. Zipf (1949), J. Vernon Henderson (1974) and Paul Krugman (1996).
[4] Imposed by Gibrat's law in Gibrat (1931)
[5] See Krugman (1996) and Duranton (2006) for details.

process and distribution of cities: classical urban growth models and random growth models. In the classical urban growth models, city growth is driven by explained city characteristics and unexplained residuals; in random growth models, the growth of city is mainly determined by the growth process of exogenous shocks. These literatures show that Zip's law could be explained by various classical urban growth models and random growth models.

Gabaix (1999*a*) proposes statistics explanation for Zipf's law. Eeckhout (2004) proposes a random growth model with economic content to explain the empirical size distribution of cities. This general equilibrium model with local externalities leads to a lognormal size distribution of cities. The crucial driving forces are the proportionate growth of exogenous productivity technological shock and local externalities in firms and household. Free labor mobility equalizes equilibrium utilities across cities, which consequently determines the growth process of cities and the asymptotically size distribution of cities.This general equilibrium model proposed in Eeckhout (2004) explains Gibrat's law in growth process and size distribution of cities. One of thedriving forcesin the modelis local externality; however there is lack of further exploration of this key driving force.

The purpose of this paper is to extend the investigation of this driving forcein explainingsize distribution of cities. Weexaminethe feature of the local externality and the relation between local externality and the resulted growth process as well as size distribution of cities; this is not in Eeckhout (2004). We briefly introduce the model ofEeckhout (2004)in Section 2.1 and explain how the model generatesZipf's lawfrom a statistics view in Section 2.2. The local externality is explored theoretically in section 3. The implication of local externality is applied in a parameter simulation in Section 4. Section 2.2, 3 and 4 are the extended work by this paper.

2. Zipf's law and a theory in Eeckhout (2004)

2.1 Theory of Eeckhout (2004)

In this section, we applya general equilibrium theory of local externalities proposed in Eeckhout (2004)to explain the empirical size distribution of cities.The driving force is a random productivity process of local economies and theperfect mobility of workers.Please see the paper for details. The only factor of production in city is labor. The productivity technological advancement of city at time t is assumed to follow the motion: $A_{it} = A_{it-1}(1+q_{i,t})$. The parameter q_{it} denotes an exogenous technology shock for each city at time *t*. This city-specific technology shock is symmetric and identically independently distributed.[6]

There is positive externality as knowledge spillover, $a_{+}(N_{it})$, where N_{it} is size of city and $a'_{+}(N_{it}) > 0$which denotes the positive external effect. There is negative externality as congestion costs, $a_{-}(N_{it})$, where $a'_{-}(N_{it}) < 0$ is the negative external effect. The output per worker in city

6 See Eeckhout (2004) for complete theory.

*i*consist of city's productivity technology and the local externalities:

$$y_{it} = A_{it} a_{+}(N_{it}) a_{-}(N_{it}) = A_{it} \mathrm{E}(N_{it}) \tag{1}$$

Where $\mathrm{E}(N_{it}) = a_{+}(N_{it}) a_{-}(N_{it})$ denotes the net local externalities.

Free mobility of workers equalizes equilibrium utilities across cities. It implies that:

$$A_{it} \cdot \mathrm{E}(N_{it}) = K \tag{2}$$

Let the local externalities be power functions:

$a_{+}(N_{it}) = N^{a}{}_{it}$ and $a_{-}(N_{it}) = N^{-b}{}_{it}$

$$\mathrm{E}(N_{it}) = N_{it}{}^{a} N_{it}{}^{-b} = N_{it}{}^{a-b} = N_{it}{}^{e} \tag{3}$$

The output per workerbecomes

$$Y_{it} = A_{it} N_{it}{}^{a-b} = A_{it} N_{it}{}^{e}, \tag{4}$$

$$\ln(Y_{it}) = \ln(A_{it}) + a\ln(N_{it}) - b\ln(N_{it}),$$

where parameter *a* denotes positive local externality as knowledge spillover; positive agglomeration economies increases output with elasticity *a* with respect to city size;parameter *b* denotes negative local externality from congestion as congestioncosts which decrease output with elasticity *b*with respect to city population;we define parameter *e*asthe size elasticity of productionin the local externalitywhich is the net effect of positive and negative externalities.Let the negative externality denote congestion cost. The larger the congestion costs, the smaller the local externality.

After normalizing equalized equilibrium utility to unity, the equilibrium size of city iscomposed of technology shock and local externality:

$$N_{it} = A_{it}{}^{k} = [(1+q_{it})A_{it-1}]^{k} = (1+q_{it})^{k} N_{it-1} \tag{5}$$

where $k = \frac{-1}{a-b} = -1/e$.

$$\ln N_{it} = k \ln A_{it} \approx \ln N_{it-1} + kq_{it}$$

$$\ln N_{it-1} + kq_{it} = \ln N_{i0} + k\sum_{t=1}^{T} q_{it}$$

When the technology shock is small enough:

$$\ln N_{it} = \ln N_{i0} + k\sum_{t=1}^{T} q_{it} \tag{6}$$

where the parameter *k* is a function of the size elasticity of production which is assumed to be a constant. The exogenous technology shock is q_{it} identically independently distributed as in Gibrat's law (Gibrat, 1931). By the central limit theorem, after ***t*** period of time, is asymptotically normally distributed, and the size distribution of N_{it} city becomes lognormal.[7]

From equation (5), we have

$$dN_{it}/dq_{it} > 0, \quad if\ e < 0$$

$$dN_{it}/dq_{it} < 0, \quad if\ e > 0$$

Larger technology shocks will lead to larger cities if the size elasticity of production is negative.

[7] The city-specific technology shock is symmetric and identically independently distributed and small .

The size elasticity of production is tending to be negative if the congestioncost from congestion is very large. On the other hand, larger shocks will lead to smaller cities if the size elasticity of production is positive. The size elasticity of production is tending to be positive if the congestion cost from congestion is very small.

The local externality in Eeckhout (2004) is assumed to be negative. The positive externality is required to be less than the negative externality to prevent an ever increasing city size. On the other hand, if the local externality is positive, the limiting distribution of city sizes would become extremely unequal, all population will concentrate in one largest city due to the advantage of positive agglomeration effect. It is crucial to fix the number of cities with a negative local size effect, otherwise workers will move to new places given the disadvantage of agglomeration and result in no cities since dispersion force always dominates agglomeration force.

2.2 Zipf's law

Let $(1+q_{it})=\varepsilon_{it}$ in equation (5) to get

$$N_{it}=(1+q_{it})^k N_{it-1}=\varepsilon_{it}{}^k N_{it-1}. \qquad (7)$$

Assume the average normalized size stay constant, which implies that

$$E[\varepsilon]=\int_0^\infty \varepsilon f(\varepsilon)d\varepsilon=1. \qquad (8)$$

Let $H_t(N)$ be the tail distribution of city sizes at time t which denotes the share of cities with population size higher than N at time t.[8]

$$H_t(N)=P(N_t>N)=P(\varepsilon_t{}^k N_{t-1}>N)=E[I_{N_{t-1}>N/\varepsilon_t^k}]$$
$$=E[E[I_{N_{t-1}>N/\varepsilon_t^k}|\varepsilon_t]=\int_0^\infty H_{t-1}\left(\frac{N}{\varepsilon^k}\right)f(\varepsilon)d\varepsilon \qquad (9)$$

At the steady state:

$$H(N)=\int_0^\infty H\left(\frac{N}{\varepsilon^k}\right)f(\varepsilon)d\varepsilon \qquad (10)$$

A distribution as Zipf's law type, $H(N)=c/N$, satisfies this steady state equation.[9]

The above shows that Zipf's law satisfies the tail distribution of city size in the steady state provided the size elasticity of productionis constant. An imposed lower boundcan obtain a steady state. Zipf's law can be a steady state tail distribution of the derived proportionate growth of cities arising from a theory of local externalitiesin Eeckhout (2004).Lognormal distribution of city sizes is not a steady state and its variance keeps increasing. Lower bound of city size allows for the existence of a steady state and prevents the distribution from ever-widening. Without a lower bound, proportionate growth lead to a lognormal distribution. With a lower bound, a lognormal distribution will turn to a Pareto distribution which fits Zipf's law.

8 In the motion equation, I_A denotes function for set *A*. The expectations are over all random variables $N_t, N_{t-1}, \varepsilon_t$.
9 See Gabaix (1999a)

3. Local externality

In equation (3), parameter *e* is the size elasticity of production in the local externality. It denotes the net local externality that net agglomeration economies changes output with elasticity *e* with respect to city size. It is the net result from positive local externality such as knowledge spillover and the negative local externality such as congestion cost. The larger the congestion cost the smaller the net local externality.

In section 2, the size elasticity of production is a constant, a negative net local externality will lead to domination of dispersion forces; on the other hand, a positive net local externality will result in only one largest city in the region. The fact that region with only one largest city or with completely dispersed populations are two extreme cases which are not in reality. This suggests that initially agglomeration force may dominate as the population increases and dispersion force will dominateeventually due to increasing congestion cost. The indirect utility function of city may be a concave and non-monotonic function of population; it is eventually diminishing with city size. This proposes that size elasticity of production may varied by size of cityra ther than to be a constant, $e(N_{it})$. Let both positive and negative local extern alities be functions of city size, the net local externality becomes:

$$\mathrm{E}(N_{it}) = a_{+}(N_{it})a_{-}(N_{it}) = N_{it}^{\ e(N_{it})} \qquad (11)$$

The output per worker in equation (4) and the motion of city size in equation (6) become

,

$$Y_{it} = A_{it}N_{it}^{\ e(N_{it})} \qquad (12)$$

$$\ln N_{it} = \ln N_{i0} - (1/e(N_{it}))\sum_{t=1}^{T} q_{it} \qquad (13)$$

In this case, the motion of city size shows that the growth rate of city size varied by city size. This implies that growth of city size is not proportionate. Consequently, the central limit theorem and identically independently distributed technology shock condition cannot be applied to generate lognormal size distribution of city as in previous case. The size distribution of cities depends on the attribute of size elasticity of production in the local externality. In this case, the theory of local economies and the mobility of workers cannot explain the empirical size distribution of cities.

$k(N_{it}) = -1/e(N_{it}) = k_{it}$.

Equation (7) becomes

$$N_{it} = (\varepsilon_{it})^{-1/e_{it}} N_{it-1} = \varepsilon_{it}^{\ k_{it}} N_{it-1} \qquad (14)$$

The tail distribution of city sizes in equation (9) becomes:

$$H_t(N) = P(N_t > N) = P(\varepsilon_t^{k_t} N_{t-1} > N) = \int_0^{\infty} H_{t-1}\left(\frac{N}{\varepsilon^{k_t}}\right) f(\varepsilon)d\varepsilon$$

The steady state distribution is conditional on the form of local externalities. Whether Zipf's law could satisfythe steady state equationor not is conditional on the local externality.

Let the size elasticity of production bea linear function of city size:

$e_{it} = e(N_{it}) = e_1 - e_2 N_{it}$, $e = 0$ when $N_{it} = e_1 / e_2$

The indirect utility function becomes:

$$v(N_{it}) = h(A_{it} \cdot \mathrm{E}(N_{it}))^{\alpha} = h(A_{it} \cdot N_{it}^{e_1 - e_2 N_{it}})^{\alpha} \quad (15)$$

$$dv_{it} / dN_{it} = (e_1 - e_2 N_{it})\alpha h(A_{it} \cdot N_{it}^{e_{it}})^{\alpha-1} N_{it}^{e_{it}-1} = e_{it}\alpha h A_{it}^{\alpha-1} N_{it}^{e_{it}\alpha-1}$$

$$dv_{it} / dN_{it} > 0, \quad if\ e_{it} > 0$$

$$dv_{it} / dN_{it} < 0, \quad if\ e_{it} < 0$$

The maximized utility rises as the city population increasesgiven a positive size elasticity of production; on the contrary, the maximized utility decreases as the city population reducesgiven a negative size elasticity of production.

$$d^2 v_{it} / dN_{it}^{\ 2} = e_{it}\alpha h[e_{it}(\alpha-1)(A_{it} \cdot N_{it}^{e_{it}})^{\alpha-2} N_{it}^{e_{it}-1} + (e_{it}-1)(A_{it} \cdot N_{it}^{e_{it}})^{\alpha-1} N_{it}^{e_{it}-2}]$$

It is allowed that the indirect utility function of city be a concave and non-monotonic function of population.

$$d^2 v_{it} / dN_{it}^{\ 2} < 0, \quad if\ e_{it} > 0$$

$$d^2 v_{it} / dN_{it}^{\ 2} < 0, \quad if\ e_{it} < 0$$

After normalizing equalized equilibrium utility to unity, the equilibrium size of city becomes:

$$N_{it} = (1+q_{it})^{-1/e_{it}} N_{it-1} = (1+q_{it})^{k_{it}} N_{it-1} \quad (16)$$

where $k_{it} = -1/e_{it} = 1/(e_2 N_{it} - e_1) \propto 1/N_{it}$.

The growth process of city size cannot be reduced to a growth process with random growth rate, and therefore Gibrat's law cannot be applied.

$$\ln N_{it} = \ln N_{i0} - (1/e(N_{it}))\sum_{t=1}^{T} q_{it} = \ln N_{i0} - (1/(e_1 - e_2 N_{it}))\sum_{t=1}^{T} q_{it}$$

It shows that the central limit theorem and asymptotically normally distributedproduction technology shock cannot derive lognormal distribution of size of city N_{it} when size elasticity of production is varied by city population.

The driving force of the equilibrium theory in Eeckhout (2004) to explain the empirical size distribution of cities is mainly depending on a random productivity process and free mobility of workers. A random productivity process could result in proportionate growth of city size only if the size elasticity of productionis constant.

4. Congestion cost and the level of concentration

The size elasticity of production, parameter *e* in equation (4),is composed of the positive local externalities as knowledge spillover and the negative local externalities as congestion and transport cost. It is a net effect of positive and negative externalities. In the theory, the size elasticity of production is crucial in determining motion and resulting size distribution of cities. Change of congestion cost (parameter *b* in equation (4)) affects the growth process of cities, and consequently the size distribution of cities.

We simulate the growth of city population based on the model in Eeckhout (2004) to examine the relation between the size elasticity of productionand the resulted city sizes distribution.The equilibrium city size(equation (5)) is determined by city-specific technology shock and local externalities. In the simulation, technology shock and positive local externalities are exogenous. The technology shock is symmetric and identically independently distributed. The equilibrium city sizes and the growth of cities are endogenously determined in the model given various negative local externality denoted as transport cost in the size elasticity of production

Further, the Gini coefficient is applied to measure concentration of populationacross cities. Thegrowths of size of cities are simulated and the corresponding Gini coefficient of the resulting size distribution of cities is estimated.[10]

The simulated result is in Figure 1. Larger value of the Gini coefficient represents more concentrate among cities; on the contrary, smaller value of Gini denotes a more evenly distributed cities population across cities. The size of cities is identicalif the Gini coefficient is 0, and the size of cities is perfectly unequal if the Gini coefficient is 1. Figure 1 shows the relation between congestion cost and the corresponding estimated Gini coefficient. The trend in figure shows that the larger the congestion cost, the smaller the estimated Gini. An increase of congestion costmaylead to more evenly distributed cities population; on the other hand, a decrease of congestion cost increases the advantage of agglomerationwhich mayraise the size of large cities in the region; the degree of inequality of city size will increase.

[10] Gini index = $2\Phi(\sigma/\sqrt{2}) - 1$, where $\Phi(x)$ is the standard normal distribution with $\Phi(x) = \text{Prob}(X < x)$. (Cowell, 1995)

5. Conclusion

In this paper, a theory of local externalities in Eeckhout (2004) is applied to explain the empirically verified Zipf's law in the upper tail of size distribution of cities, and the driving forcein Eeckhout (2004) is extended in explaining size distribution of cities.We examine the feature of the local externality and the relation between local externality and the resulted growth process as well as size distribution of cities; this is not discussed in Eeckhout (2004). This paper provides an extensionof local externality and finds the theory could generate a lognormal size distribution of cities and Zipf's law in its tail only if the size elasticity of production is a constant.This result shows the condition of the theory in explaining the empirical size distribution of cities. We also finds that the theory implies that larger shocks lead to larger cities when the size elasticity of production is negative; larger productivity shocks lead to bigger cities if the congesting cost dominates the net local externalities. Moreover, an increase of congestion cost will lead to more evenly distributed cities.

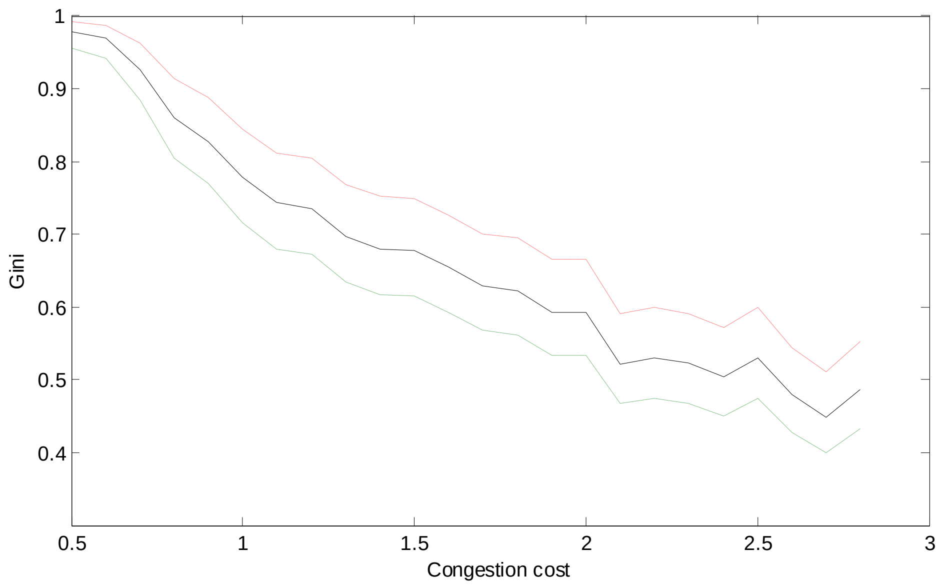

Figure 1. Congestion cost and the estimated Gini coefficient with 95% confidence interval

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